Lecture 4. Recall Thm 3 and the pf. up to compact support.

$\Rightarrow u(z)=0$ for large $|z_2|^2+\dots+|z_n|^2$.

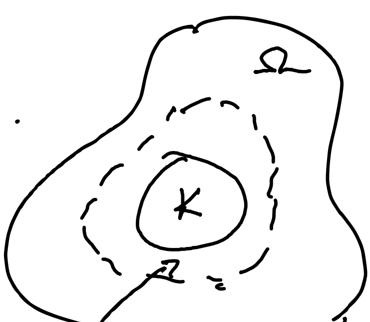
But u is holomorphic outside the supp f

$\Rightarrow u(z)$ is 0 (by uniqueness) on the unbounded component of $\mathbb{C}^n \setminus \operatorname{supp} f$ $\boxtimes$

Rem. As we saw in the proof, if $\operatorname{supp} f = K$ then $\operatorname{supp} u$ is contained in the complement of the unbdd component of K.

Hartogs' Thm. Let $\Omega \subseteq \mathbb{C}^n$ domain, $n \geq 2$, $K \subset\subset \Omega$ s.t. $\Omega \setminus K$ is connected. Then, for any $u : \Omega \setminus K \to \mathbb{C}$ holom., there is $U : \Omega \to \mathbb{C}$ holom. s.t. $u = U|_{\Omega \setminus K}$

Pf.



$K \subseteq \operatorname{supp} \varphi \subset\subset \Omega$

Let $\varphi \in C_c^\infty(\Omega)$ ← w/ compact support

s.t. $\varphi \equiv 1$ on K.

Consider $v = (1-\varphi)u$, which can be viewed as a $\mathcal{C}^\infty$ function on Ω s.t. $v = 0$ on K and $v = u$ on $\Omega \setminus \operatorname{supp}\varphi$. Of course, v is not holom. so we try to make it so. Look for U of the form $U = v + u_0$. For U to be holom: we must have

u_0 ← TBD

$$0 = \bar\partial U = \bar\partial v + \bar\partial u_0, \quad \text{i.e.}$$

$\bar\partial u_0 = -\bar\partial v = f$. Since $f = -\bar\partial v$, we have $\bar\partial f = 0$. Moreover, since $v = u$ (thus, holom.) on $\Omega \setminus \operatorname{supp}\varphi$, we have $\operatorname{supp} f \subseteq \operatorname{supp}\varphi \Rightarrow$ we can view f as a $\mathcal{C}^\infty$ $(0,1)$ form in $\mathbb{C}^n$ w/ compact supports <u>in</u> $\operatorname{supp}\varphi$. By Thm 3, the equation $\bar\partial u_0 = f$ has a solution $u_0 \in \mathcal{C}^\infty$ w/ compact support (since $n \geq 2$) and $u_0 = 0$ in the unbdd component of $\mathbb{C}^n \setminus \operatorname{supp}\varphi$ (by Rem following Thm).

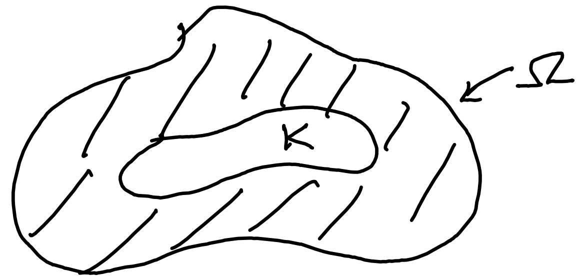
Thus, U is holom. in Ω. We must show it is equal to u in $\Omega \setminus K$. But $u_0 = 0$ on the unbounded component G of $\mathbb{C}^n \setminus \mathrm{supp}\,\varphi$. The boundary of this unbounded component G is in $\Omega \setminus K \Rightarrow$ the open subset $G' :=$ $(\Omega \setminus K) \cap G \neq \emptyset$, and on this set

$u_0 = 0$ and $V = u \Rightarrow U = u$ on $G' \subseteq \Omega \setminus K$.

But since both are holomorphic in $\Omega \setminus K$, which is connected, $U = u$ in $\Omega \setminus K$ by the uniqueness mentioned in last lecture. $\boxtimes$

<u>Rem.</u> Note that the compactness of the support of the solution to $\bar{\partial} u_0 = f$ is essential to the proof. Therefore the lack of compactness for $n=1$ explains why the same result obviously fails in $n=1$.

Hartogs' Thm is an example of domains ($\Omega \setminus K$) such that all holomorphic functions in the domain extend to a larger one (all of Ω).



But there are other extension phenomena as well.

<u>Power Series and Domains of Convergence.</u>

Let's consider a power series (centered at 0)

$$\sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha$$

<u>Def.</u> The <u>Domain of Convergence</u> D is the set of $z_0 \in \mathbb{C}^n$ s.t. P.S. converges absolutely for all z in an open nbhd of z_0. (Thus, by def., D <u>open</u>).

<u>Note.</u> In $\mathbb{C}$, D is always an open disk, whose radius is the radius of convergence.

<u>Thm</u> Assume $D \neq \emptyset$. The following hold.

(i) $D =$ interior of $\{z_0 : |a_\alpha z^\alpha| \le C$ for all α and some $C\}$ and the convergence in D is normal ($\Rightarrow$ P.S. conv. to a holom. fcn in D).

(ii) D is <u>Reinhardt</u>, i.e. invariant under $t = (t_1, \dots, t_n) \to (e^{it_1} z_1, \dots, e^{it_n} z_n)$.

(iii) If $D^* = \{\xi \in \mathbb{R}^n : (e^{\xi_1}, \dots, e^{\xi_n}) \in D\}$ open (essentially $z \in D \Leftrightarrow \log|z| \in D^*$), then $D^* \subseteq \mathbb{R}^n$ is <u>convex</u>, $\xi \in D^* \Rightarrow \eta \in D^*$ provided that $\eta_j \le \xi_j$, $j = 1, \dots, n$, and $z \in D \Leftrightarrow |z_j| \le e^{\xi_j}$, $j = 1, \dots, n$, for some $\xi \in D^*$.

<u>Def</u> Reinhardt domains Ω satisfying prop's in (iii) are called <u>logarithmically convex</u>.